Summary of Results in $N = 1$ Supersymmetric $SU(2)$ Gauge Theories [1]

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ABSTRACT

We summarize some results in $4d$, $N = 1$ supersymmetric $SU(2)$ gauge theories: the exact effective superpotentials, the vacuum structure, and the exact effective Abelian couplings for arbitrary bare masses and Yukawa couplings.

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1 The Main Result

The new results in this summary are based on some of the results in refs. [1, 2]. We consider $N = 1$ supersymmetric $SU(2)$ gauge theories in four dimensions, with any possible content of matter superfields, such that the theory is either one-loop asymptotic free or conformal. This allows the introduction of $2N_f$ matter supermultiplets in the fundamental representation, Q_i^a, $i = 1, ..., 2N_f$, N_A supermultiplets in the adjoint representation, Φ_α^{ab}, $\alpha = 1, ..., N_A$, and $N_{3/2}$ supermultiplets in the spin $3/2$ representation, Ψ. Here a, b are fundamental representation indices, and $\Phi^{ab} = \Phi^{ba}$ (we present Ψ in a schematic form as we shall not use it much). The numbers N_f, N_A and $N_{3/2}$ are limited by the condition:

$$b_1 = 6 - N_f - 2N_A - 5N_{3/2} \geq 0, \tag{1.1}$$

where $-b_1$ is the one-loop coefficient of the gauge coupling beta-function.

The main result of this talk is the following: the effective superpotential of an (asymptotically free or conformal) $N = 1$ supersymmetric $SU(2)$ gauge theory, with $2N_f$ doublets and N_A triplets (and $N_{3/2}$ quartets) is

$$
\begin{aligned}
W_{N_f, N_A}(M, X, Z, N_{3/2}) &= -\delta_{N_{3/2}, 0}(4 - b_1) \left\{ \Lambda^{-b_1} \mathrm{Pf}_{2N_f} X \left[\det\nolimits_{N_A}(\Gamma_{\alpha\beta}) \right]^2 \right\}^{1/(4 - b_1)} \\
&+ \mathrm{Tr}_{N_A} \tilde{m} M + \frac{1}{2} \mathrm{Tr}_{2N_f} m X + \frac{1}{\sqrt{2}} \mathrm{Tr}_{2N_f} \lambda^\alpha Z_\alpha + \delta_{N_{3/2}, 1} g U,
\end{aligned} \tag{1.2}
$$

where

$$\Gamma_{\alpha\beta}(M, X, Z) = M_{\alpha\beta} + \mathrm{Tr}_{2N_f}(Z_\alpha X^{-1} Z_\beta X^{-1}). \tag{1.3}$$

The first term in (1.2) is the exact (dynamically generated) nonperturbative superpotential[6], and the other terms are the tree-level superpotential. Λ is the dynamically generated scale, while $\tilde{m}_{\alpha\beta}$, m_{ij} and λ_{ij}^α are the bare masses and Yukawa couplings, respectively ($\tilde{m}_{\alpha\beta} = \tilde{m}_{\beta\alpha}$, $m_{ij} = -m_{ji}$, $\lambda_{ij}^\alpha = \lambda_{ji}^\alpha$). The gauge singlets, X, M, Z, U, are given in terms of the $N = 1$ superfield doublets, Q^a, the triplets, Φ^{ab}, and the quartets, Ψ, as follows:

$$
\begin{aligned}
X_{ij} &= Q_{ia} Q_j^a, & a &= 1, 2, & i, j &= 1, ..., 2N_f, \\
M_{\alpha\beta} &= \Phi_{\alpha b}^a \Phi_{\beta a}^b, & \alpha, \beta &= 1, ..., N_A, & a, b &= 1, 2, \\
Z_{ij}^\alpha &= Q_{ia} \Phi_{\alpha b}^a Q_j^b, & & & U &= \Psi^4.
\end{aligned} \tag{1.4}
$$

Here, the a, b indices are raised and lowered with an ϵ_{ab} tensor. The gauge-invariant superfields, X_{ij}, may be considered as a mixture of $SU(2)$ "mesons" and "baryons", while the gauge-invariant superfields, Z_{ij}^α, may be considered as a mixture of $SU(2)$ "meson-like" and "baryon-like" operators.

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[6]Integrating in the "glueball" field $S = -W_\alpha^2$, whose source is $\log \Lambda^{b_1}$, gives the nonperturbative superpotential:

$$W(S, M, X, Z) = S \left[\log \left(\frac{\Lambda^{b_1} S^{4 - b_1}}{\mathrm{Pf} X (\det \Gamma)^2} \right) - (4 - b_1) \right].$$

Equation (1.2) is a universal representation of the superpotential for all infra-red non-trivial theories; all the physics we shall discuss (and beyond) is in (1.2). In particular, all the symmetries and quantum numbers of the various parameters are already embodied in W_{N_f,N_A}. The nonperturbative superpotential is derived in refs. [1, 2] by an "integrating in" procedure, following refs. [3, 4]. The details can be found in ref. [2], and will not be presented here. Instead, we list the main results concerning each of the theories, $N_f, N_A, N_{3/2}$, case by case.

2 $b_1 = 6$: $N_f = N_A = N_{3/2} = 0$

This is a pure $N = 1$ supersymmetric $SU(2)$ gauge theory. The nonperturbative effective superpotential is[7]

$$W_{0,0} = \pm 2\Lambda^3. \tag{2.1}$$

This theory was considered before [5]. The superpotential in eq. (2.1) can be derived by integrating out the matter of any of the other theories; it is non-zero due to gluino condensation. The "$\pm$" in (2.1) comes from the square-root, appearing on the braces in (1.2), when $b_1 = 6$; it corresponds, physically, to the two quantum vacua of a pure $N = 1$ supersymmetric $SU(2)$ gauge theory.

3 $b_1 = 5$: $N_f = 1$, $N_A = N_{3/2} = 0$

There is one case with $b_1 = 5$, namely, $SU(2)$ with one flavor. This theory was considered before [5]; it is a particular case of $SU(N_c)$ with $N_f = N_c - 1$. The superpotential is

$$W_{1,0} = \frac{\Lambda^5}{X} + mX, \tag{3.1}$$

where X and m are defined by: $X_{ij} \equiv X\epsilon_{ij}$, $m_{ij} \equiv -m\epsilon_{ij}$. The nonperturbative part of $W_{1,0}$ is proportional to the one instanton action. The vacuum degeneracy of the classical low-energy effective theory is lifted quantum mechanically; from eq. (3.1) we see that, in the massless case, there is no vacuum at all.

4 $b_1 = 4$

There are two cases with $b_1 = 4$: either $N_f = 2$, or $N_A = 1$. In both cases, the nonperturbative superpotential vanishes and, in addition, there is a constraint[8].

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[7]This can be read from eq. (1.2) by setting $\mathrm{Tr}_{2N_f}(\cdot) = 0$, $\det_{2N_f}(\cdot) = 1$ (for example, $\mathrm{Pf}X = 1$, $\Gamma = M$) when $N_f = 0$, and $\mathrm{Tr}_{N_A}(\cdot) = 0$, $\det_{N_A}(\cdot) = 1$ (for example, $\det \Gamma = 1$) when $N_A = 0$; this will also be used later.

[8]This is reflected in eq. (1.2) by the vanishing of the coefficient $(4 - b_1)$ in front of the braces, leading to $W = 0$, and the singular power $1/(4-b_1)$ on the braces, when $b_1 = 4$, which signals the existence of a constraint.

4.1 $N_f = 2$, $N_A = N_{3/2} = 0$

The nonperturbative superpotential vanishes

$$W_{2,0}^{non-per.} = 0, \tag{4.1}$$

and by the integrating in procedure we also get the quantum constraint:

$$\mathrm{Pf} X = \Lambda^4. \tag{4.2}$$

This theory was considered before [5]; it is a particular case of $SU(N_c)$ with $N_f = N_c$. At the classical limit, $\Lambda \to 0$, the quantum constraint collapses into the classical constraint, $\mathrm{Pf} X = 0$.

4.2 $N_f = 0$, $N_A = 1$, $N_{3/2} = 0$

The massless $N_A = 1$ case is a pure $SU(2)$, $N = 2$ supersymmetric Yang-Mills theory. This model was considered in detail in ref. [6]. The nonperturbative superpotential vanishes

$$W_{0,1}^{non-per.} = 0, \tag{4.3}$$

and by the integrating in procedure we also get the quantum constraint:

$$M = \pm \Lambda^2. \tag{4.4}$$

This result can be understood because the starting point of the integrating in procedure is a pure $N = 1$ supersymmetric Yang-Mills theory. Therefore, it leads us to the points at the verge of confinement in the moduli space. These are the two singular points in the M moduli space of the theory; they are due to massless monopoles or dyons. Such excitations are not constructed out of the elementary degrees of freedom and, therefore, there is no trace for them in W. (This situation is different if $N_f \neq 0$, $N_A = 1$; in this case, monopoles are different manifestations of the elementary degrees of freedom.)

5 $b_1 = 3$

There are two cases with $b_1 = 3$: either $N_f = 3$, or $N_A = N_f = 1$. In both cases, for vanishing bare parameters in (1.2), the semi-classical limit, $\Lambda \to 0$, imposes the classical constraints, given by the equations of motion: $\partial W = 0$; however, quantum corrections remove the constraints.

5.1 $N_f = 3$, $N_A = N_{3/2} = 0$

The superpotential is

$$W_{3,0} = -\frac{\mathrm{Pf} X}{\Lambda^3} + \frac{1}{2}\mathrm{Tr} m X. \tag{5.1}$$

This theory was considered before [5]; it is a particular case of $SU(N_c)$ with $N_f = N_c + 1$. In the massless case, the equations $\partial_X W = 0$ give the classical constraints; in particular, the superpotential is proportional to a classical constraint: $\text{Pf} X = 0$. The negative power of Λ, in eq. (5.1) with $m = 0$, indicates that small values of Λ imply a semi-classical limit for which the classical constraints are imposed.

5.2 $N_f = 1$, $N_A = 1$, $N_{3/2} = 0$

In this case, the superpotential in (1.2) reads

$$W_{1,1} = -\frac{\text{Pf} X}{\Lambda^3}\Gamma^2 + \tilde{m}M + \frac{1}{2}\text{Tr} mX + \frac{1}{\sqrt{2}}\text{Tr}\lambda Z. \tag{5.2}$$

Here m, X are antisymmetric 2×2 matrices, λ, Z are symmetric 2×2 matrices and

$$\Gamma = M + \text{Tr}(ZX^{-1})^2. \tag{5.3}$$

This superpotential was found before in ref. [7]. To find the quantum vacua, we solve the equations: $\partial_M W = \partial_X W = \partial_Z W = 0$. Let us discuss some properties of this theory:

- The equations $\partial W = 0$ can be re-organized into the singularity conditions of an elliptic curve:

$$y^2 = x^3 + ax^2 + bx + c \tag{5.4}$$

  (and some other equations), where the coefficients a, b, c are functions of only the field M, the scale Λ, the bare quark masses, m, and Yukawa couplings, λ. Explicitly,

$$a = -M, \qquad b = \frac{\Lambda^3}{4}\text{Pf} m, \qquad c = -\frac{\alpha}{16}, \tag{5.5}$$

  where

$$\alpha = \frac{\Lambda^6}{4}\det\lambda. \tag{5.6}$$

- The parameter x, in the elliptic curve (5.4), is given in terms of the composite field:

$$x \equiv \frac{1}{2}\Gamma. \tag{5.7}$$

- $W_{1,1}$ has $2 + N_f = 3$ vacua, namely, the three singularities of the elliptic curve in (5.4), (5.5). These are the three solutions, $M(x)$, of the equations: $y^2 = \partial y^2/\partial x = 0$; the solutions for X, Z are given by the other equations of motion.

- The *3 quantum vacua* are the vacua of the theory in the *Higgs-confinement phase* [8].

- Phase transition points to the *Coulomb branch* are at $X = 0 \Leftrightarrow \tilde{m} = 0$. Therefore, we conclude that the elliptic curve defines the *effective Abelian coupling*, $\tau(M, \Lambda, m, \lambda)$, in the Coulomb branch.

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- On the subspace of bare parameters, where the theory has an enhanced $N = 2$ supersymmetry, the result in eq. (5.5) coincides with the result in [9] for $N_f = 1$.

- In the massless case, there is a $Z_{4-N_f} = Z_3$ global symmetry acting on the moduli space.

- When the masses and Yukawa couplings approach zero, all the 3 singularities collapse to the origin. Such a point might be interpreted as a new scale-invariant theory [5]. As before, the negative power of Λ, in eq. (5.2) with $\tilde{m} = m = \lambda = 0$, indicates that small values of Λ imply a semi-classical limit for which the classical constraint, $\Gamma = 0$, is imposed. Indeed, for vanishing bare parameters, the equations of motion are solved by any M, X, Z obeying $\Gamma = 0$.

6 $b_1 = 2$

There are three cases with $b_1 = 2$: $N_f = 4$, or $N_A = 1$, $N_f = 2$, or $N_A = 2$. In all three cases, for vanishing bare parameters in (1.2), there are extra massless degrees of freedom not included in the procedure; those are expected due to a non-Abelian conformal theory.

6.1 $N_f = 4$, $N_A = N_{3/2} = 0$

The superpotential is

$$W_{4,0} = -2\frac{(\mathrm{Pf}X)^{\frac{1}{2}}}{\Lambda} + \frac{1}{2}\mathrm{Tr}mX. \tag{6.1}$$

This theory was considered before in [5]; it is a particular case of $SU(N_c)$ with $N_f > N_c + 1$. In the massless case, the superpotential is proportional to the square-root of a classical constraint: $\mathrm{Pf}X = 0$. The branch cut at $\mathrm{Pf}X = 0$ signals the appearance of extra massless degrees of freedom at these points; those are expected in ref. [10]. Therefore, we make use of the superpotential only in the presence of masses, m, which fix the vacua away from such points.

6.2 $N_f = 2$, $N_A = 1$, $N_{3/2} = 0$

In this case, the superpotential in (1.2) reads

$$W_{2,1} = -2\frac{(\mathrm{Pf}X)^{\frac{1}{2}}}{\Lambda}\Gamma + \tilde{m}M + \frac{1}{2}\mathrm{Tr}mX + \frac{1}{\sqrt{2}}\mathrm{Tr}\lambda Z. \tag{6.2}$$

Here m, X are antisymmetric 4×4 matrices, λ, Z are symmetric 4×4 matrices and Γ is given in eq. (5.3). As in section 5.2, to find the quantum vacua, we solve the equations: $\partial W = 0$. Let us discuss some properties of this theory:

- The equations $\partial W = 0$ can be re-organized into the singularity conditions of an elliptic curve (5.4) (and some other equations), where the coefficients a, b, c are functions of only the field M, the scale Λ and the bare quark masses, m, and Yukawa couplings, λ. Explicitly [1, 2],

$$a = -M, \qquad b = -\frac{\alpha}{4} + \frac{\Lambda^2}{4}\mathrm{Pf}m, \qquad c = \frac{\alpha}{8}\left(2M + \mathrm{Tr}(\mu^2)\right), \qquad (6.3)$$

where

$$\alpha = \frac{\Lambda^4}{16}\det\lambda, \qquad \mu = \lambda^{-1}m. \qquad (6.4)$$

- As in section 5.2, the parameter x, in the elliptic curve (5.4), is given in terms of the composite field:

$$x \equiv \frac{1}{2}\Gamma. \qquad (6.5)$$

Therefore, we have identified a physical meaning of the parameter x.

- $W_{2,1}$ has $2 + N_f = 4$ vacua, namely, the four singularities of the elliptic curve in (5.4), (6.3). These are the four solutions, $M(x)$, of the equations: $y^2 = \partial y^2/\partial x = 0$; the solutions for X, Z are given by the other equations of motion.

- The *4 quantum vacua* are the vacua of the theory in the *Higgs-confinement phase.*

- Phase transition points to the *Coulomb branch* are at $X = 0 \Rightarrow \tilde{m} = 0$. Therefore, we conclude that the elliptic curve defines the *effective Abelian coupling*, $\tau(M, \Lambda, m, \lambda)$, in the Coulomb branch.

- On the subspace of bare parameters, where the theory has an enhanced $N = 2$ supersymmetry, the result in eq. (6.3) coincides with the result in [9] for $N_f = 2$.

- In the massless case, there is a $Z_{4-N_f} = Z_2$ global symmetry acting on the moduli space.

- For special values of the bare masses and Yukawa couplings, some of the 4 vacua degenerate. In some cases, it may lead to points where mutually non-local degrees of freedom are massless, similar to the situation in pure $N = 2$ supersymmetric gauge theories, considered in [11]. For example, when the masses and Yukawa couplings approach zero, all the 4 singularities collapse to the origin. Such points might be interpreted as in a *non-Abelian Coulomb phase* [5].

- The singularity at $X = 0$ (in Γ) and the branch cut at $\mathrm{Pf}X = 0$ (due to the 1/2 power in eq. (6.2)) signal the appearance of extra massless degrees of freedom at these points; those are expected similar to refs. [10, 12]. Therefore, we make use of the superpotential only in the presence of bare parameters, which fix the vacua away from such points.

6.3 $N_f = 0$, $N_A = 2$, $N_{3/2} = 0$

In this case, the superpotential in eq. (1.2) reads

$$W_{0,2} = \pm 2\frac{\det M}{\Lambda} + \text{Tr}\tilde{m}M. \tag{6.6}$$

Here $\tilde{m}$, M are 2×2 symmetric matrices, and the "$\pm$" comes from the square-root, appearing on the braces in (1.2), when $b_1 = 2$. The superpotential in eq. (6.6) is the one presented in [7, 13] on the confinement and the oblique confinement branches[9] (they are related by a discrete symmetry [5]). This theory has *two quantum vacua*; these become the phase transition points to the Coulomb branch when $\det \tilde{m} = 0$. The moduli space may also contain a non-Abelian Coulomb phase when the two singularities degenerate at $M = 0$ [7]; this happens when $\tilde{m} = 0$. At this point, the theory has extra massless degrees of freedom and, therefore, $W_{0,2}$ fails to describe the physics at $\tilde{m} = 0$. Moreover, at $\tilde{m} = 0$, the theory has other descriptions via an electric-magnetic triality [5].

7 $b_1 = 1$

There are four cases with $b_1 = 1$: $N_f = 5$, or $N_A = 1$, $N_f = 3$, or $N_A = 2$, $N_f = 1$, or $N_{3/2} = 1$.

7.1 $N_f = 5$, $N_A = N_{3/2} = 0$

The superpotential is

$$W_{5,0} = -3\frac{(\text{Pf}X)^{\frac{1}{3}}}{\Lambda^{\frac{1}{3}}} + \frac{1}{2}\text{Tr}mX. \tag{7.1}$$

This theory was considered before in [5]; it is a particular case of $SU(N_c)$ with $N_f > N_c + 1$. The discussion in section 6.1 is relevant in this case too.

7.2 $N_f = 3$, $N_A = 1$, $N_{3/2} = 0$

In this case, the superpotential in (1.2) reads

$$W_{3,1} = -3\frac{(\text{Pf}X)^{\frac{1}{3}}}{\Lambda^{\frac{1}{3}}}\Gamma^{\frac{2}{3}} + \tilde{m}M + \frac{1}{2}\text{Tr}mX + \frac{1}{\sqrt{2}}\text{Tr}\lambda Z. \tag{7.2}$$

Here m, X are antisymmetric 6×6 matrices, λ, Z are symmetric 6×6 matrices and Γ is given in eq. (5.3). Let us discuss some properties of this theory:

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[9] The fractional power $1/(4 - b_1)$ on the braces in (1.2), for any theory with $b_1 \leq 2$, may indicate a similar phenomenon, namely, the existence of confinement and oblique confinement branches of the theory, corresponding to the $4 - b_1$ phases due to the fractional power. It is plausible that, for $SU(2)$, such branches are related by a discrete symmetry.

- The equations $\partial W = 0$ can be re-organized into the singularity conditions of an elliptic curve (5.4) (and some other equations), where the coefficients a, b, c are [1, 2]

$$a = -M - \alpha, \qquad b = 2\alpha M + \frac{\alpha}{2}\mathrm{Tr}(\mu^2) + \frac{\Lambda}{4}\mathrm{Pf}m,$$

$$c = \frac{\alpha}{8}\Big(-8M^2 - 4M\mathrm{Tr}(\mu^2) - [\mathrm{Tr}(\mu^2)]^2 + 2\mathrm{Tr}(\mu^4)\Big), \qquad (7.3)$$

where

$$\alpha = \frac{\Lambda^2}{64}\det\lambda, \qquad \mu = \lambda^{-1}m. \qquad (7.4)$$

In eq. (7.3) we have shifted the quantum field M to

$$M \rightarrow M - \alpha/4. \qquad (7.5)$$

- The parameter x, in the elliptic curve (5.4), is given in terms of the composite field:

$$x \equiv \frac{1}{2}\Gamma + \frac{\alpha}{2}. \qquad (7.6)$$

Therefore, as before, we have identified a physical meaning of the parameter x.

- $W_{3,1}$ has $2 + N_f = 5$ *quantum vacua*, corresponding to the 5 singularities of the elliptic curve (5.4), (7.3); these are the vacua of the theory in the *Higgs-confinement phase*.

- From the phase transition points to the *Coulomb branch*, we conclude that the elliptic curve defines the *effective Abelian coupling*, $\tau(M, \Lambda, m, \lambda)$, for arbitrary bare masses and Yukawa couplings. As before, on the subspace of bare parameters, where the theory has $N = 2$ supersymmetry, the result in eq. (7.3) coincides with the result in [9] for $N_f = 3$.

- For special values of the bare masses and Yukawa couplings, some of the 5 vacua degenerate. In some cases, it may lead to points where mutually non-local degrees of freedom are massless, and might be interpreted as in a *non-Abelian Coulomb phase* or another new *superconformal theory* in four dimensions (see the discussion in section 6.2).

- The singularity and branch cuts in $W_{3,1}$ signal the appearance of extra massless degrees of freedom at these points. The discussion in the end of section 6.2 is relevant here too.

7.3 $N_f = 1$, $N_A = 2$, $N_{3/2} = 0$

In this case, the superpotential in (1.2) reads [2]

$$W_{1,2} = -3\frac{(\mathrm{Pf}X)^{1/3}}{\Lambda^{1/3}}(\det\Gamma)^{2/3} + \mathrm{Tr}\tilde{m}M + \frac{1}{2}\mathrm{Tr}mX + \frac{1}{\sqrt{2}}\mathrm{Tr}\lambda^\alpha Z_\alpha. \qquad (7.7)$$

Here m and X are antisymmetric 2×2 matrices, λ^α and Z_α are symmetric 2×2 matrices, $\alpha = 1, 2$, $\tilde{m}$, M are 2×2 symmetric matrices and $\Gamma_{\alpha\beta}$ is given in eq. (1.3). This theory has *3*

quantum vacua in the Higgs-confinement branch. At the phase transition points to the Coulomb branch, namely, when $\det \tilde{m} = 0 \Leftrightarrow \det M = 0$, the equations of motion can be re-organized into the singularity conditions of an elliptic curve (5.4). Explicitly, when $\tilde{m}_{22} = \tilde{m}_{12} = 0$, the coefficients a, b, c in (5.4) are [2]

$$a = -M_{22}, \qquad b = \frac{\Lambda \tilde{m}_{11}^2}{16} \mathrm{Pf} m, \qquad c = -\Big(\frac{\Lambda \tilde{m}_{11}^2}{32}\Big)^2 \det \lambda_2. \tag{7.8}$$

However, unlike the $N_A = 1$ cases, the equations $\partial W = 0$ *cannot* be re-organized into the singularity condition of an elliptic curve, in general. This result makes sense, physically, since an elliptic curve is expected to "show up" only at the phase transition points to the Coulomb branch. For special values of the bare parameters, there are points in the moduli space where (some of) the singularities degenerate; such points might be interpreted as in a non-Abelian Coulomb phase, or new superconformal theories. For more details, see ref. [2].

7.4 $N_f = N_A = 0$, $N_{3/2} = 1$

This chiral theory was shown to have $W_{0,0}^{non-per.}(N_{3/2} = 1) = 0$ [14]; perturbing it by a tree-level superpotential, $W_{tree} = gU$, where U is given in (1.4), may lead to dynamical supersymmetry breaking [14].

8 $b_1 = 0$

There are five cases with $b_1 = 0$: $N_f = 6$, or $N_A = 1$, $N_f = 4$, or $N_A = N_f = 2$, or $N_A = 3$, or $N_{3/2} = N_f = 1$. These theories have vanishing one-loop beta-functions in either conformal or infra-red free beta-functions and, therefore, will possess extra structure.

8.1 $N_f = 6$, $N_A = N_{3/2} = 0$

This theory is a particular case of $SU(N_c)$ with $N_f = 3N_c$; the electric theory is free in the infra-red [5].[10]

8.2 $N_f = 4$, $N_A = 1$, $N_{3/2} = 0$

In this case, the superpotential in (1.2) reads

$$W_{4,1} = -4 \frac{(\mathrm{Pf} X)^{\frac{1}{4}}}{\Lambda^{\frac{b_1}{4}}} \Gamma^{\frac{1}{2}} + \tilde{m}M + \frac{1}{2}\mathrm{Tr} mX + \frac{1}{\sqrt{2}}\mathrm{Tr}\lambda Z. \tag{8.1}$$

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[10] A related fact is that (unlike the $N_A = 1$, $N_f = 4$ case, considered next) in the (would be) superpotential, $W_{6,0} = -4\Lambda^{-b_1/4}(\mathrm{Pf} X)^{1/4} + \frac{1}{2}\mathrm{Tr} mX$, it is impossible to construct the matching "Λ^{b_1}"$\equiv f(\tau_0)$, where τ_0 is the non-Abelian gauge coupling constant, in a way that respects the global symmetries.

Here m, X are antisymmetric 8×8 matrices, λ, Z are symmetric 8×8 matrices, Γ is given in eq. (5.3) and

$$\Lambda^{b_1} \equiv 16\alpha(\tau_0)^{1/2}(\det \lambda)^{-1/2}, \tag{8.2}$$

where $\alpha(\tau_0)$ will be presented in eq. (8.4). Let us discuss some properties of this theory:

- The equations $\partial W = 0$ can be re-organized into the singularity conditions of an elliptic curve (5.4) (and some other equations), where the coefficients a, b, c are [1, 2]

$$
\begin{aligned}
a &= \frac{1}{\beta^2}\Big\{2\frac{\alpha+1}{\alpha-1}M + \frac{8}{\beta^2}\frac{\alpha}{(\alpha-1)^2}\mathrm{Tr}(\mu^2)\Big\}, \\
b &= \frac{1}{\beta^4}\Big\{-16\frac{\alpha}{(\alpha-1)^2}M^2 + \frac{32}{\beta^2}\frac{\alpha(\alpha+1)}{(\alpha-1)^3}M\mathrm{Tr}(\mu^2) \\
&\quad - \frac{8}{\beta^4}\frac{\alpha}{(\alpha-1)^2}\big[(\mathrm{Tr}(\mu^2))^2 - 2\mathrm{Tr}(\mu^4)\big] + \frac{4}{\beta^4}\frac{(\alpha+1)\Lambda^{b_1}}{(\alpha-1)^2}\mathrm{Pf}m\Big\}, \\
c &= \frac{1}{\beta^6}\Big\{-32\frac{\alpha(\alpha+1)}{(\alpha-1)^3}M^3 + \frac{32}{\beta^2}\frac{\alpha(\alpha+1)^2}{(\alpha-1)^4}M^2\mathrm{Tr}(\mu^2) \\
&\quad + M\Big[-\frac{16}{\beta^4}\frac{\alpha(\alpha+1)}{(\alpha-1)^3}\big((\mathrm{Tr}(\mu^2))^2 - 2\mathrm{Tr}(\mu^4)\big) + \frac{32}{\beta^4}\frac{\alpha\Lambda^{b_1}}{(\alpha-1)^3}\mathrm{Pf}m\Big] \\
&\quad - \frac{32}{\beta^6}\frac{\alpha}{(\alpha-1)^2}\Big[\mathrm{Tr}(\mu^2)\mathrm{Tr}(\mu^4) - \frac{1}{6}(\mathrm{Tr}(\mu^2))^3 - \frac{4}{3}\mathrm{Tr}(\mu^6)\Big]\Big\}.
\end{aligned}
\tag{8.3}
$$

Here $\mu = \lambda^{-1}m$ and α, β are functions of τ_0, the non-Abelian gauge coupling constant; comparison with ref. [9] gives

$$\alpha(\tau_0) \equiv \frac{``\Lambda^{2b_1}"}{256}\det\lambda = \left(\frac{\theta_2^2 - \theta_3^2}{\theta_2^2 + \theta_3^2}\right)^2, \qquad \beta(\tau_0) = \frac{\sqrt{2}}{\theta_2\theta_3}, \tag{8.4}$$

where

$$\theta_2(\tau_0) = \sum_{n\in Z}(-1)^n e^{\pi i \tau_0 n^2}, \qquad \theta_3(\tau_0) = \sum_{n\in Z} e^{\pi i \tau_0 n^2}, \qquad \tau_0 = \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}. \tag{8.5}$$

In eq. (8.3) we have shifted the quantum field M to

$$M \to \beta^2 M - \alpha\mathrm{Tr}\mu^2/(\alpha-1). \tag{8.6}$$

- The parameter x, in the elliptic curve (5.4), is given in terms of the composite field:

$$x \equiv \frac{1}{\beta^4}\Big[\Gamma - \frac{4\alpha}{(\alpha-1)^2}\mathrm{Tr}\mu^2\Big]. \tag{8.7}$$

- $W_{4,1}$ has $2 + N_f = 6$ *quantum vacua*, corresponding to the 6 singularities of the elliptic curve (5.4), (8.3); these are the vacua of the theory in the *Higgs-confinement phase*.

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- As before, from the phase transition points to the *Coulomb branch*, we conclude that the elliptic curve defines the *effective Abelian coupling*, $\tau(M, \Lambda, m, \lambda)$, for arbitrary bare masses and Yukawa couplings. On the subspace of bare parameters, where the theory has $N = 2$ supersymmetry, the result in eq. (8.3) coincides with the result in [9] for $N_f = 4$.

- The discussion in the end of sections 6.2 and 7.2 is relevant here too.

- We can get the results for $N_A = 1$, $N_f < 4$, by integrating out flavors.

- In all the $N_A = 1$, $N_f \neq 0$ cases we *derived* the result that τ is a section of an $SL(2, \mathbf{Z})$ bundle over the moduli space and over the parameters space of bare masses and Yukawa couplings (since τ is a modular parameter of a torus).

8.3 $N_f = 2$, $N_A = 2$, $N_{3/2} = 0$

It was argued that this theory is infra-red free [2].[11]

8.4 $N_f = 0$, $N_A = 3$, $N_{3/2} = 0$

In this case, the superpotential in eq. (1.2) reads

$$W_{0,3} = -4\frac{(\det M)^{\frac{1}{2}}}{\Lambda^{\frac{b_1}{4}}} + \mathrm{Tr}\tilde{m}M. \tag{8.8}$$

Here $\tilde{m}$, M are 3×3 symmetric matrices. The superpotential (8.8) equals to the tree-level superpotential, $W_{tree} = \lambda \det \Phi$, where, schematically, $\det \Phi \sim \epsilon\Phi\Phi\Phi \sim (\det M)^{1/2}$ is the (anti-symmetric) coupling of the three triplets, Φ_α. This result coincides with the one derived in [13]. Therefore, we identify the matching "$\Lambda^{-b_1/4}$" $\equiv \lambda f(\tau_0)$, which respects the global symmetries. In the massless case, this theory flows to an $N = 4$ supersymmetric Yang-Mills fixed point.

8.5 $N_f = 1$, $N_A = 0$, $N_{3/2} = 1$

It was argued that this theory is infra-red free [1].

9 More Results

We have summarized some old and new results in $N = 1$ supersymmetric $SU(2)$ gauge theories. More new results, along the lines of this investigation, were derived in [2, 15, 16]. In ref. [2], we have derived some results in $N = 1$ supersymmetric $SU(N_c)$ gauge theories, $N_c >$

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[11]A related fact is that (unlike the $N_A = 1$, $N_f = 4$ case) it is impossible to construct a matching, "Λ^{b_1}" $\equiv \alpha(\tau_0)f(\lambda^\alpha)$, in a way that respects the global symmetries.

2, with N_A matter supermultiplets in the adjoint representation, N_f supermultiplets in the fundamental representation and N_f supermultiplets in the anti-fundamental representation. More properties of $SU(N_c)$ supersymmetric gauge theories were studied and will be reported in [15]. Moreover, preliminary results in $SU(2) \times SU(2)$ supersymmetric gauge theories, with matter supermultiplets in the $(1,3)$ and $(2,2)^n$ representations, were reported in this symposium by S. Forste, and will appear in [16].

Acknowledgements

References

[1] S. Elitzur, A. Forge, A. Giveon and E. Rabinovici, hep-th/9504080, Phys. Lett. **B353** (1995) 79.

[2] S. Elitzur, A. Forge, A. Giveon and E. Rabinovici, hep-th/9509130. To appear in Nucl. Phys. B.

[3] K. Intriligator, R.G. Leigh and N. Seiberg, Phys. Rev. **D50** (1994) 1092.

[4] K. Intriligator, Phys. Lett. **B336** (1994) 409.

[5] For a review, see: K. Intriligator and N. Seiberg, hep-th/9509066, and references therein.

[6] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19.

[7] K. Intriligator and N. Seiberg, Nucl. Phys. **B431** (1994) 551.

[8] T. Banks and E. Rabinovici, Nucl. Phys. **B160** (1979) 349; E. Fradkin and S. Shenker, Phys. Rev. **D19** (1979) 3682.

[9] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484.

[10] N. Seiberg, Nucl. Phys. **B435** (1995) 129.

[11] P. Argyres and M. Douglas, Nucl. Phys. **B448** (1995) 93.

[12] D. Kutasov, Phys. Lett. **B351** (1995) 230.

[13] K. Intriligator and N. Seiberg, Nucl. Phys. **B444** (1995) 125.

[14] K. Intriligator, N. Seiberg and S.H. Shenker, Phys. Lett. **B342** (1995) 152.

[15] S. Elitzur, A. Forge, A. Giveon and E. Rabinovici, to appear.

[16] S. Elitzur, S. Forste, A. Giveon and E. Rabinovici, to appear.